

LOCALLY CONFORMAL SYMPLECTIC NILMANIFOLDS WITH NO LOCALLY CONFORMAL KÄHLER METRICS

GIOVANNI BAZZONI AND JUAN CARLOS MARRERO

ABSTRACT. We obtain an example of a compact locally conformal symplectic nilmanifold which admits no locally conformal Kähler metrics. This gives a new positive answer to a question raised by L. Ornea and M. Verbitsky.

1. INTRODUCTION

In the seminal paper [17], Gray and Hervella studied almost Hermitian structures on even dimensional manifolds. Recall that an almost Hermitian structure on a manifold M of dimension $2n$ consists of a Riemannian metric g and a compatible almost complex structure J , that is, an endomorphism $J: TM \rightarrow TM$ with $J^2 = -\text{Id}$, such that $g(JX, JY) = g(X, Y)$ for every $X, Y \in \mathfrak{X}(M)$. Let (g, J) be an almost Hermitian structure on M ; then one can consider the following tensors on M :

- a 2-form $\omega \in \Omega^2(M)$, the Kähler form, defined by $\omega(X, Y) = g(JX, Y)$, $X, Y \in \mathfrak{X}(M)$;
- a 1-form $\theta \in \Omega^1(M)$, the Lee form, defined by $\theta(X) = \frac{-1}{n-1} \delta \omega(JX)$, where δ is the codifferential and $X \in \mathfrak{X}(M)$.

Gray and Hervella classified almost Hermitian structures (g, J) by studying the covariant derivative of the Kähler form ω with respect to the Levi-Civita connection ∇ of g . Of particular interest for us are the following classes:

- the class of Kähler structures, for which $\nabla \omega = 0$. In this case, g is said to be a Kähler metric. Compact examples of manifolds endowed with Kähler structures are Riemann surfaces, complex tori, complex projective spaces and projective varieties; notice that the Lee form is zero on a Kähler manifold;
- the class of locally conformal Kähler (lcK from now on) structures, for which $d\omega = \omega \wedge \theta$; in this case we call g a lcK metric. A remarkable example of manifold endowed with a lcK structure is the Hopf surface. The study of lcK metrics on compact complex surfaces was undertaken in [8].

In this note we will use indifferently the terminology (locally conformal) Kähler *manifold* and (locally conformal) Kähler *structure*. For a comprehensive introduction to Kähler geometry we refer to [20]; for lcK geometry, the standard reference is [12]; see also the recent survey [32].

Kähler and locally conformal Kähler manifolds are examples of complex manifolds; this means that the almost complex structure J is integrable, hence our manifold M^{2n} is locally modelled on \mathbb{C}^n . Furthermore, on Kähler and locally conformal Kähler manifolds, any two of the tensors (g, J, ω) determine the third. One can interpret Kähler manifolds as a “degenerate” case of locally conformal Kähler manifolds. Indeed, it turns out that the Lee form of a lcK structure is closed; suppose that it is exact; one can then easily show that there is a Kähler metric in the conformal class of g . In such case, one says that the structure is globally conformal Kähler (gcK). In general, genuine lcK manifolds (those for which the Lee form is not exact) admit an open cover $\{U_\alpha\}$ such that the Lee form is exact on each U_α , hence the lcK metric is locally conformal to a Kähler metric.

2010 *Mathematics Subject Classification*. Primary: 53D05. Secondary: 53C55.

Key words and phrases. Locally conformal symplectic manifolds; locally conformal Kähler manifolds; compact nilmanifolds.

The non-metric version of Kähler manifolds are symplectic manifolds (see [25]). A manifold M^{2n} is symplectic if there exists a 2-form $\omega \in \Omega^2(M)$ such that $d\omega = 0$ and ω^n is a volume form. Clearly, the Kähler form of a Kähler structure is symplectic. It is well known that the existence of a Kähler metric on a compact manifold deeply influences its topology. In fact, suppose M is a compact Kähler manifold of dimension $2n$ and let ω be the Kähler form; then:

- the odd Betti numbers of M are even;
- the Lefschetz map $H^p(M) \rightarrow H^{2n-p}(M)$, $[\alpha] \mapsto [\alpha \wedge \omega^{n-p}]$ is an isomorphism for $0 \leq p \leq n$;
- M is formal.

For a long time, the only known examples of symplectic manifolds came from Kähler (or even projective) geometry. In [34], Thurston constructed the first example of a compact, symplectic 4-manifold which violates each of the three conditions given above. Since then, the problem of constructing compact symplectic manifolds without Kähler structures has inspired beautiful Mathematics (see, for instance, [13, 16, 24, 30]).

The non-metric version of locally conformal Kähler manifolds are locally conformal symplectic (lcs) manifolds (see for instance [3, 37]). A manifold M^{2n} , endowed with an almost symplectic structure ω (that is, ω^n is a volume form on M), is said to be lcs if there exists a closed 1-form θ such that $d\omega = \omega \wedge \theta$. When $n \geq 2$ the 1-form θ is uniquely determined and is called the Lee form. The defining condition of a lcs structure is equivalent to the existence of an open cover $\{U_\alpha\}$ of M such that the Lee form is exact on each open set U_α ; on such U_α , a suitable multiple ω' of the form ω is closed, hence ω' is a symplectic structure on U_α . Analogous to the Kähler case, the Kähler form and the Lee form of a locally conformal Kähler structure define a locally conformal symplectic structure. A lcs manifold is globally conformal symplectic (gcs) if the Lee form is exact; in this case, a multiple of ω is closed. A lcs manifold with vanishing first Betti number is automatically gcs.

The three properties of compact Kähler manifolds that we listed above constrain the topology of compact Kähler manifolds; on the other hand, the topology of symplectic manifolds is not constrained, see for instance [11]. In contrast, not much is known on the topology of compact lcK manifolds. In [12, Conjecture 2.1] it was conjectured that a compact lcK manifold which satisfies the topological restrictions of a Kähler manifold admits a (global) Kähler metric. A stronger conjecture (see [12, Conjecture 2.2]) is that a compact lcK manifold which is not gcK must have at least one Betti number which is odd. The latter conjecture was proven in the context of Vaisman structures, namely it was shown in [22, 36] that a compact Vaisman manifold has odd b_1 . Recall that Vaisman structures are a special class of lcK structures: a lcK metric g is of Vaisman type if, in the conformal class of g , there is a metric g' such that the Lee form is parallel with respect to the Levi-Civita connection of g' . However, the statement was disproven for general lcK metrics; in [29], Oeljeklaus and Toma constructed a lcK manifold with $b_1 = 2$.

As a consequence of the lack of topological obstructions, it is not easy to exclude that a manifold carries a locally conformal Kähler metric. This motivates the following problem, proposed by Ornea and Verbitsky in [32, Open Problem 1]:

Construct a compact lcs manifold which admits no lcK metrics.

In [2], Bande and Kotschick gave a method to construct examples of compact lcs manifolds which carry no lcK metrics; we describe it briefly. Suppose that M is an oriented compact manifold of dimension 3. Using some results on contact structures in dimension 3 [27], we have that M admits a contact structure and, then, the product manifold $M \times S^1$ is lcs (see [2, 37]). On the other hand, using some results on compact complex surfaces, one deduces that M may be chosen in such a way that the product manifold $M \times S^1$ admits no complex structures (for instance, if M is hyperbolic) and, in particular, no lcK structures.

The main result of this note is to present another kind of examples of compact lcs manifolds which admit no lcK metrics. More precisely, we prove:

Theorem 1. *There exists an example of a 4-dimensional compact nilmanifold which carries a locally conformal symplectic structure but no locally conformal Kähler metrics. This manifold is not a product of a 3-dimensional compact manifold and a circle.*

The example of Thurston that we mentioned above (which admits a Vaisman structure) is also a nilmanifold. This makes the parallel between the symplectic and the locally conformal symplectic case particularly transparent.

This note is organized as follows. In Section 2 we review some basics about nilmanifolds and describe briefly Thurston's example. In Section 3 we prove the main result.

Acknowledgements J. C. Marrero acknowledges support from MEC (Spain) Grants MTM2011-15725-E, MTM2012-34478 and the project of the Canary Government ProID20100210. G. Bazzoni is partially supported by MEC (Spain) Grant MTM2010-17389. The authors would like to thank Prof. M. Fernández, L. Ornea and M. Verbitsky for useful comments and remarks.

2. BASICS ON NILMANIFOLDS

Let G be a simply connected n -dimensional nilpotent Lie group. A *nilmanifold* is a homogeneous space $N = \Gamma \backslash G$ where $\Gamma \subset G$ is a lattice. Let \mathfrak{g} be the Lie algebra of G ; then \mathfrak{g} is a nilpotent Lie algebra and the exponential map $\exp: \mathfrak{g} \rightarrow G$ is a global diffeomorphism. Hence G is diffeomorphic to \mathbb{R}^n . This entails that $p: G \rightarrow N$ is a covering map and that N is the Eilenberg-MacLane space $K(\Gamma, 1)$. Tori are simple examples of nilmanifolds. A slightly less trivial example is the following: let H be the Heisenberg group, defined as

$$H = \left\{ \begin{pmatrix} 1 & y & z \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix} \mid x, y, z \in \mathbb{R} \right\}.$$

Then H is a nilpotent Lie group. A lattice $H_{\mathbb{Z}}$ in H is provided by matrices in H such that $x, y, z \in \mathbb{Z}$. Hence $H_{\mathbb{Z}} \backslash H$ is a compact nilmanifold.

There is a simple criterion for the existence of a lattice Γ in a nilpotent Lie group G :

Theorem 2 (Mal'cev, [26]). *Let G be a simply connected nilpotent Lie group with Lie algebra \mathfrak{g} . Then G admits a lattice if and only if there exists a basis of \mathfrak{g} with respect to which the structure constants are rational numbers.*

Suppose $N = \Gamma \backslash G$ is a compact nilmanifold and let us identify \mathfrak{g} with left-invariant vector fields on G . These vector fields are, in particular, invariant under the (left) action of Γ on G . We can also identify \mathfrak{g}^* , the dual of \mathfrak{g} , with left-invariant 1-forms on G . In order to describe a geometric structure on a nilmanifold N (given, for example, in terms of forms and vector fields), we can specify its “value” at the (class of) the identity element $[e] \in N$, that is, by giving a corresponding geometric structure in \mathfrak{g} .

Let $\{X_1, \dots, X_n\}$ be a basis of \mathfrak{g} and let $\{x_1, \dots, x_n\}$ denote the dual basis of \mathfrak{g}^* , defined by $x_i(X_j) = \delta_{ij}$. Dualizing the bracket, we obtain a map $d: \mathfrak{g}^* \rightarrow \wedge^2 \mathfrak{g}^*$; for $x_i \in \mathfrak{g}^*$, $dx_i(X_j, X_k) = -x_i([X_j, X_k])$. d is then extended to the whole exterior algebra $\wedge^* \mathfrak{g}^*$ by imposing the graded Leibnitz rule. We thus obtain a differential complex $(\wedge^* \mathfrak{g}^*, d)$, called Chevalley-Eilenberg complex. The property $d^2 = 0$ is equivalent to the Jacobi identity in \mathfrak{g} . The Chevalley-Eilenberg complex $(\wedge^* \mathfrak{g}^*, d)$ computes the Lie algebra cohomology of \mathfrak{g} .

Example 3. Let $N = \Gamma \backslash G$ be a nilmanifold of dimension $2n$. An invariant symplectic structure on N is described by the mean of a 2-form $\omega \in \wedge^2 \mathfrak{g}^*$ such that $d\omega = 0$ and $\omega^n \neq 0$.

Example 4. Let G denote the product $H \times \mathbb{R}$, where H is the Heisenberg group described above. It is easy to see that the Lie algebra \mathfrak{g} of G admits a basis $\{X_1, \dots, X_4\}$ with only non-zero bracket $[X_1, X_2] = -[X_2, X_1] = -X_4$. Accordingly, \mathfrak{g}^* has a basis $\{x_1, \dots, x_4\}$ with $dx_1 = dx_2 = dx_3 = 0$ and $dx_4 = x_1 \wedge x_2$. Then $\omega = x_1 \wedge x_4 + x_2 \wedge x_3 \in \wedge^2 \mathfrak{g}^*$ is closed and satisfies $\omega^2 \neq 0$. Let $\Gamma \subset G$ denote the lattice $H_{\mathbb{Z}} \times \mathbb{Z}$ and let $N = \Gamma \backslash G$ be the corresponding nilmanifold. By abuse of notation, we denote

also by ω the 2-form on N obtained from $\omega \in \wedge^2 \mathfrak{g}^*$. Then $\omega \in \Omega^2(N)$ is closed and nondegenerate, hence (N, ω) is a compact symplectic nilmanifold.

One feature of nilmanifolds, which makes them particularly tractable from a cohomological point of view, is that their de Rham cohomology is computed by the Chevalley-Eilenberg complex. More precisely,

Theorem 5 (Nomizu, [28]). *Let $N = \Gamma \backslash G$ be a compact nilmanifold. Then the natural inclusion $(\wedge^* \mathfrak{g}^*, d) \hookrightarrow (\Omega^*(N), d)$ induces an isomorphism in cohomology.*

Corollary 6. *Let $N = \Gamma \backslash G$ be a compact nilmanifold of dimension n . Then, for every $0 \leq i \leq n$, $b_i(N) = b_i(\mathfrak{g})$.*

If we apply Corollary 6 to the nilmanifold N of Example 4, we see that $b_1(N) = 3$, since 3 of the 4 generators of \mathfrak{g}^* are closed. Comparing this with the topological restrictions imposed by the existence of a Kähler metric on a compact manifold, we see immediately that N can not carry any Kähler metric. One can also easily show that N is neither Hard Lefschetz nor formal. The nilmanifold N of Example 4 is known as the Kodaira-Thurston manifold. On the one hand, it appeared in the classification of compact complex surfaces carried out by Kodaira (see [23]); on the other hand, it was the first example of a compact symplectic manifold without any Kähler structure in the celebrated paper [34] by Thurston.

As a matter of fact, compact Kähler nilmanifolds are very rare. More precisely, we have the following theorem:

Theorem 7 (Benson-Gordon, [9] and Hasegawa, [19]). *Let N be a $2n$ -dimensional compact nilmanifold endowed with a Kähler structure. Then N is diffeomorphic to the torus T^{2n} .*

In particular, Benson and Gordon proved that a compact symplectic nilmanifold which satisfies the Hard Lefschetz property must be diffeomorphic to a torus. Hasegawa showed that the same conclusion holds for a formal compact nilmanifold.

As a consequence, compact symplectic non-toral nilmanifolds are never Kähler. The problem of determining which even-dimensional nilmanifolds carry a symplectic structure is still open (see for instance [18]), but is completely solved in dimension 2, 4 and 6 (see [6]). Notice that, by Nomizu Theorem, a compact nilmanifold has a symplectic structure if and only if it has a left-invariant one.

In what follows, we shall adopt the standard notation for nilpotent Lie algebras/nilmanifolds. It is best explained by the mean of an example: $(0, 0, 0, 0, 12, 34)$ denotes a 6-dimensional nilmanifold $N = \Gamma \backslash G$ such that \mathfrak{g} has a basis $\{X_1, \dots, X_6\}$ with dual basis $\{x_1, \dots, x_6\}$ such that $dx_1 = \dots = dx_4 = 0$, $dx_5 = x_1 \wedge x_2$ and $dx_6 = x_3 \wedge x_4$, where d is the Chevalley-Eilenberg differential. Different choices of lattices in G give rise to coverings between the corresponding nilmanifolds.

We have the following result

Proposition 8 (see for instance [7, 30]). *Let $N = \Gamma \backslash G$ be a compact 4-dimensional nilmanifold. Then \mathfrak{g}^* is one of the following:*

1. $(0, 0, 0, 0)$ and N is diffeomorphic to a torus T^4
2. $(0, 0, 0, 12)$ and N is diffeomorphic to (a finite quotient of) the Kodaira-Thurston manifold
3. $(0, 0, 12, 13)$ and N is diffeomorphic to (a finite quotient of) an iterated circle bundle over T^2 .

Correspondingly, in each cases N carries a (left-invariant) symplectic structure, which we describe in terms of a basis $\{x_1, \dots, x_4\}$ of \mathfrak{g}^ :*

1. $\omega = x_1 \wedge x_2 + x_3 \wedge x_4$
2. $\omega = x_1 \wedge x_4 + x_2 \wedge x_3$
3. $\omega = x_1 \wedge x_4 + x_2 \wedge x_3$.

3. PROOF OF THE MAIN THEOREM

We prove next two propositions, of which Theorem 1 is a corollary.

Proposition 9. *Let $N = \Gamma \backslash G$ be the nilmanifold $(0,0,12,13)$. Then N is symplectic and locally conformal symplectic but does not admit any complex structure.*

Proof. A symplectic structure on N has been described explicitly in Proposition 8. In terms of the basis of \mathfrak{g}^* considered above, the locally conformal symplectic structure is described by taking the almost symplectic 2-form to be $\omega = x_1 \wedge x_3 + x_4 \wedge x_2$ and the Lee form to be $\theta = x_2$. Then $d\omega = \theta \wedge \omega$. Notice that θ is not exact, hence the structure is genuinely lcs. Nomizu Theorem implies that $b_1(N) = 2$. Assume that N admits a complex structure. By standard results in the theory of compact complex surfaces (see for instance [4, Theorem 3.1, page 144]), a complex surface with even first Betti number admits a Kähler metric. Since N is clearly not diffeomorphic to T^4 , this would give a contradiction with Theorem 7. Hence N does not carry any complex structure. \square

Proposition 10. *The nilmanifold N considered above is not the product of a compact 3-dimensional manifold and a circle.*

Proof. By Nomizu theorem, we see that $b_1(N) = 2$. Assume N is a product $M \times S^1$, where M is a compact 3-dimensional manifold. Since N is a compact nilmanifold, the fundamental group Γ of N is nilpotent and torsion-free. If $\Lambda = \pi_1(M)$, then $\Lambda \subset \Gamma$ is also nilpotent and torsion-free. In [26], Mal'cev showed that for such a group Λ there exists a real nilpotent Lie group $\Lambda_{\mathbb{R}}$ such that $P = \Lambda \backslash \Lambda_{\mathbb{R}}$ is a nilmanifold. It is well known that a compact nilmanifold has first Betti number at least 2, hence $b_1(P) \geq 2$. Now P is an aspherical manifold; in this case, $H^*(P; \mathbb{Z})$ is isomorphic to the group cohomology $H^*(\Lambda; \mathbb{Z})$ of Λ with coefficients in the trivial Λ -module \mathbb{Z} (see for instance [10, page 40]). Hence $b_1(\Lambda) \geq 2$. Now M is also an aspherical space with fundamental group Λ , hence, by the same token, $b_1(M) \geq 2$. However, by the Künneth formula applied to $N = M \times S^1$, we get $b_1(N) = 1$. Alternatively, we could apply [14, Theorem 3] directly to our manifold M . \square

Proof of Theorem 1. Assume N carries a lcK metric. Then N must be in particular a complex manifold, and this contradicts Proposition 9. By Proposition 10, N is not a product of a compact 3-manifold and a circle. \square

The lcs structure of the compact nilmanifold $(0,0,12,13)$, which was considered in the proof of Proposition 9, is of the first kind in the terminology of Vaisman [37]. A lcs manifold of the first kind is locally the product of a contact manifold with the real line [37] and the typical example of a compact lcs manifold of the first kind is the product of a compact contact manifold with the circle S^1 . However, our compact example is not globally a product manifold. This is a good motivation in order to discuss the global structure of lcs manifolds of the first kind. In fact, this topic is the subject of a work in progress [5].

An explicit realization of the compact nilmanifold $(0,0,12,13)$ is the following one. Let G be the connected simply connected Lie group $G = \mathbb{R}^4$ endowed with the multiplication

$$(x, y, z, t) \cdot (x', y', z', t') = (x + x', y + y', z + z' + yx', t + t' + zx' + y\frac{(x')^2}{2}).$$

Then, (\mathbb{R}^4, \cdot) is a nilpotent Lie group and

$$\{x_1 = dx, x_2 = dy, x_3 = dz - ydx, x_4 = dt - zdx\}$$

is a basis of left-invariant 1-forms on G . Moreover,

$$\Gamma = 2\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$$

is a co-compact discrete subgroup of G .

The proof of Theorem 1 relies heavily on the classification of compact complex surfaces. It is therefore unlikely that our argument can be extended to the higher dimensional case.

It was conjectured in [2, Conjecture 5.3] that any compact complex surface also admits a lcK structure (not necessarily compatible with the given complex structure). By what we argued so far, this

conjecture is true in the context of 4-dimensional compact nilmanifolds. The torus T^4 admits a (globally) conformally Kähler structure and the Kodaira-Thurston carries a Vaisman structure (see [33]). The nilmanifold N which we used in the proof of Theorem 1 is not complex.

4-dimensional nilmanifolds display interesting behaviors with respect to the geometric structure we are dealing with:

- the torus T^4 is Kähler;
- the Kodaira-Thurston manifold is complex, symplectic, Vaisman but not Kähler;
- the nilmanifold $(0, 0, 12, 13)$ is symplectic, locally conformal symplectic, but not complex (hence neither Kähler nor lcK).

Remark 11. *It would be interesting to find an example of a manifold which is genuinely lcs and complex but neither symplectic nor lcK. It is not hard to come up with such an example if one puts the more restrictive condition of not being of Vaisman type.*

It was conjectured in [35] that if a $2n$ -dimensional compact nilmanifold $N = \Gamma \backslash G$ carries a lcK metric, then $\mathfrak{g} = \mathfrak{h}_{2n-1} \oplus \mathbb{R}$, where \mathfrak{h}_{2n-1} is the generalized Heisenberg algebra; this is the Lie algebra with basis $\{X_1, Y_1, \dots, X_{n-1}, Y_{n-1}, Z\}$ and non-zero brackets $[X_i, Y_i] = -[Y_i, X_i] = -Z$ for $i = 1, \dots, n-1$ (notice that \mathfrak{h}_3 is the Lie algebra of the Heisenberg group). This conjecture has been proved by Sawai in [33], assuming that the complex structure of the lcK metric is left-invariant. A positive answer to this conjecture would produce immediately many examples of lcs non lcK manifolds. Vaisman structures on solvmanifolds have been considered in [21]. More generally, it was proven recently in [15] that compact homogeneous lcK structures are Vaisman (see also [1] for a non-compact extension of this result).

Remark 12. *Since the Kodaira-Thurston manifold carries a Vaisman metric, it is clear that compact locally conformal Kähler manifolds need not be formal. Can one, in some sense to be made precise, “bound” the non-formality of a compact lcK manifold (for instance, in terms of non-zero Massey products)? The study of geometric formality in the context of Vaisman metrics was tackled in [31].*

REFERENCES

- [1] D. V. Alekseevsky, V. Cortés, K. Hasegawa and Y. Kamishima, *Homogeneous locally conformally Kähler and Sasaki manifolds*, preprint, <http://arxiv.org/pdf/1403.3268.pdf>. (cited on p. 6)
- [2] G. Bande and D. Kotschick, *Contact Pairs and Locally Conformal Symplectic Structures*, Harmonic Maps and Differential Geometry, 85–98, Contemp. Math., 542, Amer. Math. Soc., Providence, RI, 2011. (cited on p. 2, 5)
- [3] A. Banyaga, *Some properties of locally conformal symplectic structures*, Comment. Math. Helv. 77 (2002) 383–398. (cited on p. 2)
- [4] W. P. Barth, K. Hulek, C. A. M. Peters and A. Van de Ven, *Compact complex surfaces*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3), 4, Springer-Verlag, Berlin, 2004. (cited on p. 5)
- [5] G. Bazzoni and J. C. Marrero, *Locally conformal symplectic manifolds of the first kind*, Work in progress. (cited on p. 5)
- [6] G. Bazzoni and V. Muñoz, *Classification of minimal algebras over any field up to dimension 6*, Trans. Amer. Math. Soc. 364 (2), 1007–1028, 2012. (cited on p. 4)
- [7] G. Bazzoni and V. Muñoz, *Manifolds which are complex and symplectic but not Kähler*, preprint, <http://arxiv.org/abs/1404.7662>. (cited on p. 4)
- [8] F. Belgun, *On the metric structure of non-Kähler complex surfaces*, Math. Ann. 317 (2000), no. 1, 1–40. (cited on p. 1)
- [9] C. Benson and C. S. Gordon, *Kähler and symplectic structures on nilmanifolds*, Topology 27, 513–518, 1988. (cited on p. 4)
- [10] K. S. Brown, *Cohomology of Groups*, GTM 87, Springer-Verlag, 1982. (cited on p. 5)
- [11] G. R. Cavalcanti, *The Lefschetz property, formality and blowing up in symplectic geometry*, Trans. Amer. Math. Soc. 359, no. 1, 333–348, 2007. (cited on p. 2)
- [12] S. Dragomir and L. Ornea, *Locally conformal Kähler geometry*, Progress in Mathematics 155, Birkhäuser Boston, 1998. (cited on p. 1, 2)
- [13] M. Fernández and V. Muñoz, *An 8-dimensional non-formal simply connected symplectic manifold*, Ann. of Math. (2) 167, no. 3, 1045–1054, 2008. (cited on p. 2)
- [14] M. Freedman, R. Hain and P. Teichner, *Betti number estimates for nilpotent groups*, Fields Medalists’ Lectures, 413–434, World. Sci. Ser. 20th Century Math., 5, World. Sci. Publ., River Edge, NJ, 1997. (cited on p. 5)
- [15] P. Gauduchon, A. Moroianu and L. Ornea, *Compact homogeneous lcK manifolds are Vaisman*, preprint, <http://arxiv.org/pdf/1312.6266.pdf>. (cited on p. 6)

- [16] R. E. Gompf, *A new construction of symplectic manifolds*, Ann. of Math. (2), 142(3):527–595, 1995. (cited on p. 2)
- [17] A. Gray and L. M. Hervella, *The sixteen classes of almost Hermitian manifolds and their linear invariants*, Ann. Mat. Pura Appl. (4), 123:35–58, 1980. (cited on p. 1)
- [18] Z.-D. Guan, *Toward a classification of compact nilmanifolds with symplectic structures*, Int. Math. Res. Not. IMRN 2010, no. 22, 4377–4384. (cited on p. 4)
- [19] K. Hasegawa, *Minimal models of nilmanifolds*, Proc. Amer. Math. Soc. **106**, no. 1, 65–71, 1989. (cited on p. 4)
- [20] D. Huybrechts, *Complex geometry. An introduction*, Universitext. Springer-Verlag, Berlin, 2005. (cited on p. 1)
- [21] H. Kasuya, *Vaisman metrics on solvmanifolds and Oeljeklaus-Toma manifolds*, Bull. Lond. Math. Soc. 45, no. 1, 15–26, 2013. (cited on p. 6)
- [22] T. Kashiwada and S. Sato, *On harmonic forms on compact locally conformal Kähler manifolds with parallel Lee form*, Ann. Fac. Sci. Kinshasa, Zaire. 6, 17–29, 1980. (cited on p. 2)
- [23] K. Kodaira, *On the structure of compact complex analytic surfaces. I*, Amer. J. Math. **86**, 751–798, 1964. (cited on p. 4)
- [24] D. McDuff, *Examples of symplectic simply connected manifolds with no Kähler structure*, J. Diff. Geom. **20**, 267–277, 1984. (cited on p. 2)
- [25] D. McDuff, D. Salamon, *Introduction to symplectic topology*, Oxford Science, 1995. (cited on p. 2)
- [26] A. Mal'cev, *On a class of homogeneous spaces*, Izv. Akad. Nauk. Armyan. SSSR Ser. Mat. 13 (1949), 201–212. (cited on p. 3, 5)
- [27] J. Martinet, *Formes de contact sur les variétés de dimension 3*, Lect. Notes in Math, 20 (9) (1971), 142–163. (cited on p. 2)
- [28] K. Nomizu, *On the cohomology of compact homogeneous space of nilpotent Lie group*, Ann. of Math. (2) 59 (1954), 531–538. (cited on p. 4)
- [29] K. Oeljeklaus and M. Toma, *Non-Kähler compact complex manifolds associated to number fields*, Ann. Inst. Fourier (Grenoble) 55, no. 1, 161–171, 2005. (cited on p. 2)
- [30] J. Oprea and A. Tralle, *Symplectic manifolds with no Kähler structure*, Lecture Notes in Mathematics, 1661, Springer-Verlag, Berlin, 1997. (cited on p. 2, 4)
- [31] L. Ornea and M. Pilca, *Remarks on the product of harmonic forms*, Pacific J. of Math. 250, no. 2, 353–363, 2011. (cited on p. 6)
- [32] L. Ornea and M. Verbitsky, *A report on locally conformally Kähler manifolds*, Harmonic Maps and Differential Geometry, 135–149, Contemp. Math., 542, Amer. Math. Soc., Providence, RI, 2011. (cited on p. 1, 2)
- [33] H. Sawai, *Locally conformal Kähler structures on compact nilmanifolds with left-invariant complex structures*, Geom. Dedicata 125 (2007), 93–101. (cited on p. 6)
- [34] W. Thurston, *Some simple examples of symplectic manifolds*, Proc. Amer. Math. Soc. 55 no. 2, 467–468, 1976. (cited on p. 2, 4)
- [35] L. Ugarte, *Hermitian structures on six-dimensional nilmanifolds*, Transform. Groups 12 (2007), no. 1, 175–202. (cited on p. 6)
- [36] I. Vaisman, *Generalized Hopf manifolds*, Geom. Dedicata, **13** (1982), 231–255. (cited on p. 2)
- [37] I. Vaisman, *Locally Conformal Symplectic Manifolds*, Internat. J. Math. & Math. Sci. **8** (3) (1985), 521–536. (cited on p. 2, 5)

FAKULTÄT FÜR MATHEMATIK, UNIVERSITÄT BIELEFELD, POSTFACH 100301, D-33501 BIELEFELD

E-mail address: `gbazzoni@math.uni-bielefeld.de`

UNIDAD ASOCIADA ULL-CSIC, GEOMETRÍA DIFERENCIAL Y MECÁNICA GEOMÉTRICA, UNIVERSIDAD DE LA LAGUNA, FACULTAD DE MATEMÁTICAS, AVDA. ASTROFÍSICO FRANCISCO SÁNCHEZ S/N. 38071, LA LAGUNA

E-mail address: `jcmarrer@ull.edu.es`